Compressible Plasma Flow over a Biased Body

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The flow field of a compressible plasma over a biased body is discussed with special emphasis on the electrical characteristics. The governing equations in various asymptotic regions are investigated. A stagnation-point probe theory, because of its great practical interest, is given in detail. An analytic current-voltage characteristic for probe potential between the floating and plasma potential is obtained for this case under the assumption of a very thin electrical sheath.

I. Introduction

RECENTLY there has been a revival of interest in using Langmuir probes as a measuring device in plasma. However, the classical low-density theory cannot be applied to high-density plasma flows, which are encountered in devices such as shock tubes and plasma arcs. Additional interest in the probe problem arises from its relation to the hypersonic aerodynamic problem, in which a plasma is generated behind the bow shock formed in front of a blunt body flying at hypersonic speeds. In the probe problem one is interested in the current-voltage characteristics, from which one hopes to obtain some information regarding the properties of the plasma. In the blunt-body aerodynamic problem, the main interest is in the distribution of charged particles around the body and any change in the heat-transfer characteristics which may occur due to the flux of charged particles to the body surface, which, in most practical situations, is at the floating potential.

A continuum theory of electrostatic probes in a static isothermal plasma was given by Su and Lam[1] for negative probe potentials above the floating potential and by Cohen[2] for moderate probe potentials (between the plasma and floating potential). In such analyses, the sheath was based on the collision-dominated diffusion equation. The limit of validity of such a description is obtained by requiring that the electrical energy gained by a charged particle during one free flight is much less than its thermal energy. It is relatively easy to show that such a criterion implies $\lambda_D \gg l$ for a very negative probe, and $(\lambda_D/r_p)^{2/3} \gg l/r_p$ for a moderately negative probe,[1] where λ_D is the electron Debye length based on the undisturbed charged particle density, l is a typical mean free path between charged and neutral particles, and r_p is the probe radius.

These continuum concepts were later extended by Lam[3] to an incompressible, isothermal flow of a weakly ionized gas for moderate surface potentials. Because of his assumptions that the gas is weakly ionized, incompressible, and isothermal, the diffusion of the charged particles to the solid surface is decoupled from the mass motion of the neutral gas. The existence of an electric field in the inviscid region was pointed out in this work. Chung[4] has tackled the Couette flow and stagnation flow of weakly ionized gases numerically. The sheath structure that he obtained checked qualitatively with that given in Refs. 1 and 2. This is not surprising since in the Couette flow there is no convective motion; the governing diffusion equation is identical to the static case, whereas in the stagnation flow the sheath was assumed to be thin and close to the solid surface, where convective motion is entirely negligible. The analysis in the inviscid region was neglected even though he noted that there was residual electric field intensity at the outer edge of the viscous boundary layer.

Previous to the work just discussed, Talbot[5] presented a stagnation-probe analysis. The continuum equation was used to describe the diffusion of mass, momentum, and energy in the viscous boundary layer, whereas within the sheath (which was assumed, a priori, to occupy a distance smaller than one mean free path from the surface) the charged particles fall freely down the potential hill.† The change in potential in the viscous layer was neglected. Even though Talbot's analysis is oversimplified, the quasi-continuum model is a more realistic one for plasmas of high ionization fraction. A complete analysis of this problem would, however, require a kinetic treatment. An approximate kinetic approach was recently suggested by Wasserstrom, Su, and Probstein.[13]

In the present paper, we shall adopt a strict continuum description. The restriction on such a continuum analysis which we have mentioned previously is the same for the present problem. It will be shown, however, that, for bodies at floating potential, the structure of the sheath does not enter the calculation of the heat-transfer and charged-particle distribution. Therefore the limitation mentioned is not relevant to the calculation of these quantities. We shall first discuss the electric field in the inviscid region for a general flow field. Since the flow characteristics in the viscous boundary layer are well known, the discussion concentrates on the diffusion of the charged particles and the accompanying electric potential distribution. It will be shown that, within the viscous layer, the diffusion is ambipolar in nature, even though the electron current is not necessarily equal to that for the ions. The potential distribution is decoupled from the system in the sense that it is determined after one has obtained the solutions for the other flow variables. The probe potential is assumed to be moderate, so that the sheath is thin and static (though with diffusion, of course). Finally, the stagnation probe is discussed in more detail, and an approximate analytic current-voltage characteristic is derived under the assumption of a very thin sheath.

II. Formulation

The governing continuum equations for the physical system to be discussed are as follows:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0 \tag{1}$$

$$\rho \frac{d}{dt}\left(\frac{n_\alpha}{\rho}\right) + \mathrm{div}(n_\alpha \mathbf{w}_\alpha) = 0 \tag{2}$$

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† The idea of putting a collision-dominated quasi-neutral solution and a collision-free sheath together was first suggested in 1936 by Davydov and Zmanovskaja.[6]

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}$$

and $\alpha = 1, 2, \ldots,$

$$\rho\frac{d\mathbf{v}}{dt} + \frac{\partial p}{\partial \mathbf{x}} - (n_i - n_e)e\frac{\partial \phi}{\partial \mathbf{x}} + \mathrm{div}\boldsymbol{\sigma}' = 0 \tag{3}$$

$$\rho\frac{d}{dt}\left(h + \frac{1}{2}v^2\right) = \frac{\partial p}{\partial t} + \mathrm{div}[\mathbf{v}\cdot\boldsymbol{\sigma}' - \mathbf{q}] + \mathbf{J}\cdot\mathbf{E} \tag{4}$$

$$\nabla^2\phi = -4\pi e(n_i - n_e) \tag{5}$$

where $\boldsymbol{\sigma}'$ is the viscous stress tensor.

Equation (1) is the over-all continuity equation whereas Eq. (2) is the continuity equation for each species. We shall consider a system with three species: neutrals, ions, and electrons. The species equations that need to be considered will therefore be for the ions and electrons only. Equations (3–5) are the momentum, energy, and Poisson equations, respectively. The subscripts i and e stand for ions and electrons, respectively, and all other symbols have their usual meaning. The dissipative fluxes $(n_\alpha\mathbf{w}_\alpha, \boldsymbol{\sigma}', \mathbf{q})$ are given by[7]

$$\boldsymbol{\Gamma}_i = n_i\mathbf{w}_i = -\rho D_i\left[\mathrm{grad}\frac{n_i}{\rho} + \frac{e}{kT}\frac{n_i}{\rho}\,\mathrm{grad}\phi\right] \tag{6a}$$

$$\boldsymbol{\Gamma}_e = n_e\mathbf{w}_e = -\rho D_e\left[\mathrm{grad}\frac{n_e}{\rho} - \frac{e}{kT}\frac{n_e}{\rho}\,\mathrm{grad}\phi\right] \tag{6b}$$

$$\sigma_{ik}' = \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_j}\right) + \zeta\delta_{ik}\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_j} \tag{6c}$$

$$\mathbf{q} = -\kappa\,\mathrm{grad}T + \sum_{\alpha=1}^{3}\rho_\alpha\mathbf{w}_\alpha h_\alpha \text{ (perfect gas)}\ddagger \tag{6d}$$

The boundary conditions for ρ, $\mathbf{v}$, T are well known. In addition, we shall assume that the body is a perfect absorber of charged particles, so that on the body surface $n_\alpha/\rho = 0$ for $\alpha = i$ and e. The potential on the body is given as ϕ_p with respect to the potential far ahead of the body.

We shall assume in our subsequent analysis that the net current drawn by the body is small such that the Joule heating $\mathbf{J}\cdot\mathbf{E}$ is negligible. The enthalpy diffusion flux in the heat flux vector $\mathbf{q}$ can be simplified by writing

$$\Sigma\rho_\alpha\mathbf{w}_\alpha h_\alpha = (h_i - h_g)\rho_i\mathbf{w}_i + (h_e - h_g)\rho_e\mathbf{w}_e$$

where the subscript g stands for the neutral gas and h is the thermal enthalpy per unit mass. Since $h_i = h_g$ and $h_e \gg h_g$, we have $\Sigma\rho_\alpha\mathbf{w}_\alpha h_\alpha = h_e\rho_e\mathbf{w}_e = h_e'n_e\mathbf{w}_e$, where $h_e' = \frac{5}{2}kT$ is the thermal enthalpy per electron. We shall now assume that the flow is frozen and that electrons and ions do not recombine except on the body surface. The enthalpy h in (4) and (6d) is then taken to be thermal enthalpy only. However, in our formula for the heat transfer to the wall, h will include both the thermal enthalpy and the ionization enthalpy.

We shall normalize the quantities in Eqs. (1–5) in the following way: length by a typical body dimension r_p, velocity by a typical velocity u_δ', charged-particle density by n_δ', fluid density by ρ_δ', temperature by T_δ', total enthalpy by the typical total enthalpy per unit mass of fluid by $H_\delta' = (h + \frac{1}{2}v^2)_\delta$, viscosity coefficient by μ_δ', diffusion coefficient by the ambipolar diffusion coefficient D_a,§ and the electric potential by kT_δ'/e. Then the governing equations become

$$\partial\rho/\partial t + \mathrm{div}(\rho\mathbf{v}) = 0 \tag{1a}$$

$$\rho\frac{d}{dt}\left(\frac{n_\alpha}{\rho}\right) - \frac{1}{Re}\,\mathrm{div}\left\{\frac{\mu D_\alpha}{Sc}\left[\mathrm{grad}\left(\frac{n_\alpha}{\rho}\right) \pm \frac{1}{T}\frac{n_\alpha}{\rho}\,\mathrm{grad}\phi\right]\right\} = 0 \tag{2a}$$

where the upper sign is for ion and the lower one for electron:

$$\rho\frac{d\mathbf{v}}{dt} + \frac{\partial p}{\partial \mathbf{x}} - \frac{\beta}{\alpha}\epsilon^2(n_i - n_e)\frac{\partial\phi}{\partial \mathbf{x}} + \frac{1}{Re}\,\mathrm{div}\boldsymbol{\sigma}' = 0 \tag{3a}$$

$$\rho\frac{d}{dt}\left(h + \frac{1}{2}v^2\right) = \alpha\frac{\partial p}{\partial t} + \frac{\alpha}{Re}\,\mathrm{div}\left\{\mu\left(\mathbf{v}\cdot\boldsymbol{\sigma}' - \frac{1}{Pr}\,\mathrm{grad}\frac{v^2}{2}\right) + \frac{\mu}{Pr}\,\mathrm{grad}\left(h + \frac{v^2}{2}\right)\right\} - \frac{5}{2}\frac{\beta}{Re}\,\mathrm{div}\left\{\frac{\mu T}{LeSc}\,\mathrm{grad}\left(\frac{n_e}{\rho}\right) + \frac{\mu T}{Sc\rho}\,n_e\mathbf{w}_e\right\} \tag{4a}$$

$$\epsilon^2\nabla^2\phi = -(n_i - n_e) \tag{5a}$$

We have used the same notation as in Eqs. (1–5). Note that all of the quantities in Eqs. (1a–5a) are now dimensionless. The various parameters are defined as follows:

$$Re = \frac{\rho_\delta' u_\delta' r_p}{\mu_\delta'} \qquad Sc = \frac{\mu_0}{\rho_0 D_a} \qquad Le = \frac{\rho_0\bar{c}_p D_a}{\kappa}$$

$$Pr = \frac{\bar{c}_p\mu_0}{\kappa} \qquad \bar{c}_p = \sum_{\alpha=1}^{3}\frac{\rho_\alpha c_{p\alpha}}{\rho_0}$$

$$\epsilon^2 = \frac{\lambda_D^2}{r_p^2} = \frac{kT_\delta'}{4\pi n_\delta' e^2 r_p^2}$$

$$\alpha = \frac{u_\delta'^2}{H_\delta'} \qquad \beta = \frac{n_\delta' kT_\delta'}{\rho_\delta' u_\delta'^2}$$

where all of the quantities on the right side of each expression are dimensional; ρ_0, μ_0, κ, $\bar{c}_p$, and D_a are local physical values that they stand for.

The diffusion fluxes are now

$$n_i\mathbf{w}_i = \frac{r_p\boldsymbol{\Gamma}_i}{D_a n_\delta'} = -\rho D_i\left[\mathrm{grad}\frac{n_i}{\rho} + \frac{1}{T}\frac{n_i}{\rho}\,\mathrm{grad}\phi\right]$$

$$n_e\mathbf{w}_i = \frac{r_p\boldsymbol{\Gamma}_e}{D_a n_\delta'} = -\rho D_e\left[\mathrm{grad}\frac{n_e}{\rho} - \frac{1}{T}\frac{n_e}{\rho}\,\mathrm{grad}\phi\right]$$

It is reasonable to assume that the nondimensional parameters Sc, Pr, Le (α, β) are all of order unity in comparison with the two most important parameters Re and λ_D/r_p. In most cases of practical interest, the following inequalities are satisfied, i.e.,

$$(\lambda_D/r_p)^2 \ll 1/Re \ll 1 \tag{7}$$

Both of the preceding two parameters are associated with the relevant highest-order derivatives in our system of equations. It is therefore expected that there will be two singular perturbations in the problem, one for the viscous layer, associated with the Reynolds number Re (Prandtl boundary layer), and another for the sheath, associated with the Debye length parameter λ_D/r_p (Langmuir boundary layer). Because of the inequalities,[7] we see that the sheath is imbedded within the viscous layer. Qualitatively, we can now say that there are three distinct regions where different physical mechanisms operate:

1) Inviscid region: Diffusion of mass, momentum, and energy are relatively unimportant compared with convection. Charge neutrality is maintained.

2) Viscous layer: Convection and diffusion operate simultaneously. Charge neutrality is also maintained in this region.

3) Sheath: Here charge separation can take place. Convection is unimportant since the region is thin and adjacent to a solid surface. Diffusion and mobility of the ions and electrons are the main mechanisms in the sheath.

‡ The ion and electron temperatures T are assumed to be equal.

§ The reason for using D_a will be made clear later in the discussion of the viscous boundary layer.

Because of the assumption that the sheath region adjacent to the surface is thin, we shall automatically restrict ourselves to a moderate probe potential. If the potential is strong enough, the sheath can be thick and the problem of convection within the sheath has to be properly taken into account.

III. Inviscid Region

In this region, the system of equations becomes doubly degenerate. First the Laplacian in the Poisson equation is neglected on the basis of the smallness of $(\lambda_D/r_p)^2$. This gives

$$n_i = n_e + 0\left(\frac{\lambda_D{}^2}{r_p{}^2}\right) \tag{8}$$

which is the well-known quasi-neutral solution. Next we drop all dissipation terms in Eqs. (1–4), i.e.,

$$\begin{aligned} \partial\rho/\partial t + \operatorname{div}(\rho\mathbf{v}) &= 0 \\ \rho d/dt(n_\alpha/\rho) &= 0(1/Re) \\ \rho d\mathbf{v}/dt + \partial p/\partial\mathbf{x} &= 0(1/Re) \\ \rho(d/dt)(h + \tfrac{1}{2}v^2) - 2p/\partial t &= 0(1/Re) \end{aligned} \tag{9}$$

To this order of accuracy, the density ρ, mass velocity $\mathbf{v}$, and fluid enthalpy h, as well as the charged-particle density $n = n_i = n_e$, are determined by this degenerate set of equations. However, any information regarding the electric field is lost from the system. This lost information can be recovered by subtracting the two species continuity equations (2a) (annihilation of the dominant terms), i.e.,

$$\operatorname{div}\left\{\frac{\mu}{Sc}\left[(D_i - D_e)\operatorname{grad}\left(\frac{n}{\rho}\right) + \frac{D_i + D_e}{T}\frac{n}{\rho}\operatorname{grad}\phi\right]\right\} = 0 \tag{10}$$

With n, ρ, and T determined from Eq. (9), we can calculate the electric potential in the inviscid region by means of Eq. (10). It is obvious that the validity of (10) is independent of the large Reynolds number assumption. It is valid as long as quasi-neutrality is maintained.¶ Moreover, within the sheath, even though $n_i \neq n_e$, since the mass velocity is small (of the order of the ratio of the sheath thickness to the viscous layer thickness), Eq. (10) is still approximately valid.** Note that, after multiplying by e, the quantity within the brackets in (10) is the conduction current. Thus we conclude from (10) that the conduction current through a closed surface in the flow field is zero. Within the viscous and sheath layers the flux through a surface normal to the wall is negligible, so that we have constancy of the conduction current density through these layers, i.e.,

$$(D_i - D_e)\frac{\partial}{\partial y_1}\left(\frac{n}{\rho}\right) + \frac{D_i + D_e}{T}\frac{n}{\rho}\frac{\partial\phi}{\partial y_1} = \frac{CSc}{\mu} \tag{11}$$

Where C is dimensionless, it relates to the dimensional conduction current $J = e(\Gamma_e - \Gamma_i)$ by

$$C(x) = [J(x)/e](Re/n_\delta' u_\delta')$$

and y_1 is the coordinate normal to the surface.

Equation (11) evaluated at the outer edge of the viscous layer furnishes us a boundary condition for Eq. (10), i.e.,

$$\left.\frac{\partial\phi}{\partial y_1}\right|_\delta = \frac{\rho T}{n(D_i + D_e)}\left[\frac{CSc}{\mu} - (D_i - D_e)\frac{\partial}{\partial y_1}\left(\frac{n}{\rho}\right)\right]\Bigg|_\delta \tag{12}$$

To determine the potential in the inviscid region uniquely, we also require $\phi \rightarrow 0$ at infinity.††

In special cases such as stagnation-point flow, flow over a flat plate, and the end wall problem in a shock tube, the quantities n, ρ, and T are approximately constant in the inviscid region. In this case Eq. (10) and the boundary condition Eq. (12) are greatly simplified, so that the equation for the potential and the boundary conditions becomes

$$\nabla^2\phi = 0 \tag{13a}$$

$$\left.\frac{\partial\phi}{\partial y_1}\right|_\delta = \left.\frac{\rho TCSc}{n\mu(D_i + D_e)}\right|_\delta \tag{13b}$$

$$\phi \rightarrow 0 \text{ at infinity} \tag{13c}$$

Equation (13a) had been shown by Lam[3] to be appropriate for the incompressible fluids.

The conduction current density J collected by the body is still an unknown constant that must be determined by the boundary conditions specified on the body surface. However, at the floating potential, J is by definition zero. It follows then from (13a) and (13b) that in the inviscid region $\phi(\mathbf{x}) \equiv 0$. It is seen that the reversal of the polarity of the electric field in the inviscid region occurs at the floating potential, in contrast to the plasma potential in the no-flow case.

IV. Viscous Layer

Within the viscous boundary layer, the dissipative terms (with the gradient in the direction normal to the wall) are of the same order of magnitude as the convective terms. However, quasi-neutrality is still a good approximation. In this layer we let $y_1 = Re^{-1/2}y$, $v_1 = Re^{-1/2}v$, where y_1, v_1 are the unstretched coordinate and velocity normal to the surface. The governing equations after such transformation are

$$\left.\begin{aligned} &\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0 \\ &\rho\frac{d}{dt}\left(\frac{n_i}{\rho}\right) = \frac{\partial}{\partial y}\left\{\frac{\mu D_i}{Sc}\left[\frac{\partial}{\partial y}\left(\frac{n_i}{\rho}\right) + \frac{1}{T}\frac{n_1}{\rho}\frac{\partial\phi}{\partial y}\right]\right\} \\ &\rho\frac{d}{dt}\left(\frac{n_e}{\rho}\right) = \frac{\partial}{\partial y}\left\{\frac{\mu D_e}{Sc}\left[\frac{\partial}{\partial y}\left(\frac{n_e}{\rho}\right) - \frac{1}{T}\frac{n_e}{\rho}\frac{\partial\phi}{\partial y}\right]\right\} \\ &\rho\frac{du}{dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad \frac{\partial p}{\partial y} = 0 \\ &\rho\frac{d}{dt}\left(h + \frac{u^2}{2}\right) = \alpha\frac{\partial p}{\partial t} + \alpha\frac{\partial}{\partial y}\left\{\mu\left(1 - \frac{1}{Pr}\right)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) + \frac{\mu}{Pr}\frac{\partial}{\partial y}\left(h + \frac{u^2}{2}\right)\right\} - \\ &\qquad \frac{5}{2}\,\beta\frac{\partial}{\partial y}\left\{\frac{\mu T}{LeSc}\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + \frac{\mu T}{Sc\rho}n_e\mathbf{w}_{ey}\right\} \\ &n_i = n_e + 0\left(\frac{\lambda_D{}^2}{r_p{}^2}Re\right) \end{aligned}\right\} \tag{14}$$

where w_{ey} is the electron diffusion velocity in the direction normal to the wall.

It was pointed out in the last section that Eq. (11) is valid within the viscous boundary layer. Such an equation gives the relation between the charged-particle distribution and the electric potential within the layer. As in the inviscid region, it is a great simplification that the solution for the density distribution of charged particles can be determined at first independently of the electric potential. We shall see that this is in general true, provided that quasi-neutrality is valid.

¶ We have assumed $(\lambda_D/r_p)^2 \ll 1/Re$.

** Equation (10) is approximately valid within a time-independent sheath if the convection there is negligible. Since the convective velocity on the wall is zero, the convection within the sheath is of the order of the thickness of the sheath.

†† Equation (10) is an elliptic second-order linear partial differential equation. The boundary conditions we have specified uniquely define a solution.

The second of Eqs. (14) can be integrated to give

$$\frac{Sc}{\mu D_i}\int^y \rho \frac{d}{dt}\left(\frac{n}{\rho}\right)dy = \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + \frac{1}{T}\frac{n}{\rho}\frac{\partial\phi}{\partial y} \tag{14a}$$

Similarly, for the electron continuity equation,

$$\frac{Sc}{\mu D_e}\int^y \rho \frac{d}{dt}\left(\frac{n}{\rho}\right)dy = \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) - \frac{1}{T}\frac{n}{\rho}\frac{\partial\phi}{\partial y} \tag{14b}$$

From the foregoing equations we obtain

$$\rho\frac{d}{dt}\left(\frac{n}{\rho}\right) = \frac{\partial}{\partial y}\left[\frac{2}{1/D_i + 1/D_e}\frac{\mu}{Sc}\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right)\right] = \frac{\partial}{\partial y}\left[\frac{\rho\mu}{Sc}\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right)\right] = \frac{\partial}{\partial y}\left[-\frac{\mu}{Sc}\frac{n_i w_{iy}}{\rho}\right] = \frac{\partial}{\partial y}\left[-\frac{\mu}{Sc}\frac{n_e w_{ey}}{\rho}\right] \tag{15}$$

where $2/(1/D_i + 1/D_e) = 1$ because of our normalization.

We conclude from Eq. (15) that

$$\frac{n_i w_{iy}}{\rho} = -D_i\left[\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + \frac{1}{T}\frac{n}{\rho}\frac{\partial\phi}{\partial y}\right] = -\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + A\frac{\mu}{Sc} \tag{16}$$

$$\frac{n_e w_{ey}}{\rho} = -D_e\left[\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) - \frac{1}{T}\frac{n}{\rho}\frac{\partial\phi}{\partial y}\right] = -\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + B\frac{\mu}{Sc} \tag{17}$$

where A and B are two related arbitrary constants. It is seen from Eqs. (16) and (17) that the diffusion in the viscous layer is essentially characterized by ambipolar diffusion.‡‡ However, we must emphasize that the ion and electron currents are the same only when the body is at the floating potential. In view of Eq. (15), we see that the Schmidt number introduced earlier should be based on the ambipolar diffusion coefficient. Since the latter is of the same order of magnitude as D_i, we conclude that the thickness of the diffusion layers (both for electrons and ions) is of the same order of magnitude as the viscous momentum layer, i.e., of order $Re^{-1/2}$.

The constants A and B can be determined at the outer edge of the viscous layer. We find

$$B = \frac{\mu}{Sc}\left[(1 - D_e)\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + \frac{D_e}{T}\frac{n}{\rho}\frac{\delta\phi}{\delta y}\right] \tag{18}$$

In the special cases such as stagnation-point flow, flat-plate flow, or the end wall of a shock tube, $\partial/\partial y(n/\rho)|_\delta = 0$; then

$$B = \frac{\mu}{Sc}\frac{D_e}{T}\frac{n}{\rho}\frac{\partial\phi}{\partial y}\bigg|_\delta = \frac{D_e}{D_e + D_i}\frac{C(x)}{Re^{1/2}} \tag{19}$$

Similarly,

$$A = -[D_i/(D_i + D_e)][C(x)/Re^{1/2}] \tag{20}$$

These are essentially the currents due to the mobilities of electrons and ions. At the floating potential, we see from Eqs. (19) and (20) that $A = B = 0$.

The governing equations within the viscous layer can now be simplified as follows:

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{v}) = 0$$

$$\rho\frac{d}{dt}\left(\frac{n}{\rho}\right) = \frac{\partial}{\partial y}\left[\frac{\mu}{Sc}\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right)\right]$$

$$\rho\frac{du}{dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad \frac{\partial p}{\partial y} = 0$$

[*Equation (21) continued at top of next column.*]

$$\rho\frac{d}{dt}\left(h + \frac{u^2}{2}\right) = \alpha\frac{\partial p}{\partial t} + \alpha\frac{\partial}{\partial y}\left\{\mu\left[1 - \frac{1}{Pr}\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right] + \frac{\mu}{Pr}\frac{\partial}{\partial y}\left(h + \frac{u^2}{2}\right)\right\} - \frac{\beta}{k}\frac{\delta}{\delta y}\left\{\frac{5}{2}kT \times \left[\frac{\mu}{Sc}\left(\frac{1}{Le} - 1\right)\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + B\right]\right\} \tag{21}$$

The electric potential is obtained from Eq. (11) after this set of equations is solved for ρ, n, $\mathbf{v}$, and T.

Except for the constant B term in the energy equations, Eqs. (21) are the usual boundary-layer equations for a frozen dissociated gas.[8, 9] In order to make use of the usual similarity transformations for the compressible boundary layer, it is more convenient to write Eqs. (21) in their unstretched dimensional form. To do this we first drop the factor α, β/k, and then replace μ/Sc by ρD_a. The corresponding expression for the constant B term in the energy equation will be $(n_\delta' u_\delta' B/Re^{1/2})(\partial/\partial y_1)(\frac{5}{2}\,kT)$, where y_1 is the unstretched dimensional coordinate.

We now introduce the following similarity transformation:

$$\xi(x) = \int_0^x \rho_\delta u_\delta \mu_\delta r_b^{2j} dx$$
$$\eta(x) = \frac{\rho_\delta u_\delta r_b^j}{(2\xi)^{1/2}}\int_0^{y_1}\frac{\rho}{\rho_\delta}dy \tag{22}$$

where $j = 0, 1$ for steady two- and three-dimensional flows, respectively. ρ_δ, u_δ, μ_δ are functions of x alone and will be denoted as the physical value they stand for along the edge of the viscous layer, and $r_b(x)$ defines the contour of the body.

Equations (21) §§ then reduce to

$$\left.\begin{aligned}
&(Nf'')' + ff'' = \frac{2\xi}{u_\delta}\frac{du}{d\xi}\left[f'^2 - \frac{\rho_\delta}{\rho}\right] \\
&\left(\frac{N}{Sc}z'\right)' + fz' = \frac{2\xi f'z}{(n/\rho)_\delta}\frac{d(n/\rho)_\delta}{d\xi} \\
&\left(\frac{N}{Pr}g'\right) + fg' = \frac{2\xi f'g}{H_\delta}\frac{dH_\delta}{d\xi} + \frac{u_\delta^2}{H_\delta}\left[\left(\frac{1}{Pr} - 1\right)Nf'f''\right]' + \left[\frac{N}{Sc}\left(\frac{1}{Le} - 1\right)\frac{(\frac{5}{2})kT(n/\rho)_\delta}{H_\delta}z'\right]' - \frac{n_\delta' u_\delta' B}{Re^{1/2}}\frac{(2\xi)^{1/2}}{\rho_\delta\mu_\delta u_\delta r_0^j}\frac{\partial}{\partial\eta}\left(\frac{5}{2}kT\right)
\end{aligned}\right\} \tag{23}$$

where

$$N = \frac{\rho\mu}{\rho_\delta\mu_\delta} \qquad f' - \frac{u}{u_\delta} \qquad f = \int_0^\eta \frac{u}{u_\delta}d\eta$$

$$z = \frac{n/\rho}{(n/\rho)_\delta} \qquad g = \frac{h + u^2/2}{(h + u/2)_\delta} = \frac{h + u^2/2}{H_\delta}$$

$$Sc = \mu/\rho D_a \qquad Le = \rho\bar{c}_p D_a/\kappa\text{¶¶}$$

and the prime on f, g, z indicates differentiation with respect to η. Equations (23) reduce to the ordinary differential equations if all of the terms on the right sides of them are functions of η alone. For more detailed discussion on the conditions of similarity, see Ref. 8.

We recall that Eqs. (23) are obtained under the assumption of quasi-neutrality. Such an assumption will break down inside the sheath near the wall, where we expect a rapid

‡‡ Equation (15) was first obtained by Chung for stagnation-point flow.

§§ Before applying Eq. (22) to Eq. (21), the latter must first be put into dimensional form.

¶¶ These are identical to our previous definition. For weakly ionized gas it can be shown that both Sc and Le are of order unity.[10] For a fully ionized gas one has to use some mixture rule for the transport coefficient; see Ref. 11, for example.

variation in potential. Since we shall demonstrate later that the sheath is thin, it is clear that the change in density, velocity, and temperature across the sheath will be very small. If we require Eqs. (23) to satisfy the boundary conditions given on the wall, the error introduced in the solutions for density, velocity, and temperature will be of order of the sheath thickness. The difficulty, however, arises when we try to calculate the potential distribution by means of Eq. (11) based on a charged-particle density distribution obtained in the foregoing fashion. In the first place, the current J cannot be determined. Second, and more important, the electric field becomes infinite at the edge of the sheath. This suggests that, although the velocity and temperature distributions are decoupled from the system within the sheath, the charged-particle density and potential distribution have to be solved simultaneously. However, in the problem of a blunt body at floating potential ($J = 0$), one is interested only in the charged-particle distribution around the body and the charged-particle fluxes to the wall. As long as the sheath is thin, even though it may be collision-dominated or collision-free, the charged-particle distribution and the heat flux can be determined with an error of order of sheath thickness by letting Eqs. (23) satisfy the boundary conditions given on the wall. We give the heat transfer to a floating wall as follows:

$$q_w = -\frac{k}{\bar{c}_p}\frac{\partial}{\partial y}\left(h + \frac{u^2}{2}\right) + h_e\left(\frac{k}{\bar{c}_p}\frac{\partial C_e}{\partial y} - \rho D_a \frac{\partial C_e}{\partial y}\right) - \rho D_a \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right)h^{(0)} = -\frac{\partial \eta}{\partial y}\frac{k}{\bar{c}_p}H_\delta\left\{g'(0) + \frac{\frac{5}{2}n\,kT}{\rho_\delta H_\delta}\left[Le\left(1 + \frac{2h^{(0)}}{5kT}\right) - 1\right]z'(0)\right\} \quad (24)$$

where $h^{(0)}$ is the ionization energy per electron ion pair. For the case $h^{(0)} \gg (\frac{5}{2})\,kT$,

$$q_w = -\frac{\partial \eta}{\partial y}\frac{k}{\bar{c}_p}H_\delta\left\{g'(0) + \frac{n_\delta h^{(0)}}{\rho_\delta H_\delta}\,Le\,z'(0)\right\} \quad (24a)$$

The zero arguments of g and z in Eqs. (24) and (24a) are referred to the edge of the sheath, which in the floating case can be identified as the wall.

Near the solid surface, $f \sim \eta^2$, $f' \sim \eta$, $z \sim \eta$, so that the second equation in (23) is reduced to $(N/Scz')' \simeq 0$. Thus

$$z' = C_1 = \text{const} \quad (25)$$

or

$$\frac{1}{\rho}\frac{dn}{dy} \equiv C_1\frac{dy}{d\eta} + \frac{n}{\rho^2}\frac{d\rho}{dy}$$

If we assume the density variation across the viscous boundary layer to be order one* relative to the two expansion parameters in our problem (i.e., λ_D/r_p and Re), then $C_1\,d\eta/dy$ is of order unity, whereas the second term on the right side of Eq. (25) is of the order of n or n_s/n_δ, where n_s is the number density of the charged particles at the edge of the sheath and n_δ is that at the outer edge of the viscous layer. Thus the size of the second term on the side of Eq. (25) is of the order of the thickness of the sheath relative to the viscous boundary-layer coordinate, since the charge density on the wall is assumed to be zero. The thickness of the sheath relative to the viscous-layer coordinate is, as we shall see shortly, of the order of $(\lambda_D/r_p)^{2/3}Re^{1/3}$. For the shock tubes we have here in the Fluid Mechanics Laboratory at Massachusetts Institute of Technology, $\lambda_D/r_p \sim 10^{-6}$ and $Re \sim 10^3 - 10^4$. Thus $(\lambda_D/r_p)^{2/3}\,Re^{1/3} \sim 10^{-3}$, and we shall neglect the second one:

$$dn/dy = F = \text{const} \quad (25a)$$

This behavior of the quasi-neutral solution of the charge density is exactly like that in the static case. We shall identify the constant F with Γ_i' and Γ_e' as follows:

$$\frac{dn}{dy} = -\frac{1}{2}\left(\frac{\Gamma_e'}{D_e'} + \frac{\Gamma_e'}{D_i'}\right)\frac{r_p}{n_\delta'} \quad (26)$$

where n, y are nondimensional and the latter is a stretched coordinate, whereas D_e', D_i' are dimensional.

The corresponding behavior of electric field is obtained from Eq. (11) with $J/e = Re^{1/2}\,(\Gamma_e' - \Gamma_i')$:

$$\frac{d\phi}{dy} = \frac{1}{2}\left(\frac{\Gamma_e'}{D_e'} - \frac{\Gamma_i'}{D_i'}\right)\frac{r_pT_w}{n_\delta'}\,\frac{1}{n} \quad (27)$$

The constants Γ_e', Γ_i' (they are constant within the sheath; see next section) have to be determined by the boundary conditions on the wall through an analysis of the sheath.

V. Charge Separation Sheath

Since we shall consider the case of moderate potential, the charge separation sheath can be assumed to be thin.[1] The analysis within the sheath will then be similar to that given by Cohen.[2] The appropriate transformation will be

$$y_1 = Re^{-1/2}y = Re^{-1/6}(\lambda_D/r_p)^{2/3}\xi \quad (28)$$

$$n_{i,\,e} = (\lambda_D{}^2Re/r_p{}^2)^{1/3}\bar{n}_{i,\,e} \quad (29)$$

With these transformations, it can be shown that the total fluid density, mass velocity, and temperature stay constant throughout the sheath layer, whereas the equations of species and Poisson's equation are

$$\frac{d\bar{n}_{i,\,e}}{d\xi} \pm \frac{\bar{n}_{i,\,e}}{T_w}\frac{d\phi}{d\xi} = I_{i,e}$$
$$\frac{d^2\phi}{d\xi^2} = -(\bar{n}_i - \bar{n}_e) \quad (30)$$

where the upper sign is for ion and the lower one for the electron. The solution of Eq. (30) which goes into (26) and (27) as $\xi \to \infty$ and which has $n_i = n_e = 0$, $\phi = \phi_p$ on the wall was numerically integrated in Cohen's work.[2]

From the second half of Eq. (28) it is seen that the thickness of the sheath is of the order of $Re^{1/3}\,(\lambda_D/r_p)^{2/3}$ in the viscous boundary-layer coordinate. Its value in the unstretched (natural) scale is $Re^{-1/6}\,(\lambda_D/r_p)^{2/3}$.

In the remainder of this section, we shall consider the construction of the current-voltage characteristic. The probe potential can be written as follows:

$$-\phi_p = \int_0^{\delta_s{}^+}\left(\frac{d\phi}{dy_1}\right)_s dy_1 + \int_{\delta_s{}^+}^{\delta_v{}^+}\left(\frac{d\phi}{dy_1}\right)_v dy_1 + \int_{\delta_v{}^+}^{\infty} d\mathbf{r}\cdot\nabla\phi_I \quad (31)$$

where $1 \gg \delta_s{}^+ \gg \delta_s$ (δ_s = thickness of sheath) and $1 \gg \delta_v{}^+ \gg \delta_v$ (δ_v = thickness of viscous layer).

A complete current-voltage characteristic will require the solution of the electric field in all three regions. Even though we have shown the different simplifications available in all three regions, the resulting equations in general still defy a closed-form solution. It should be noted, however, that there is a singularity at the joining place of the viscous and sheath layers. From Eq. (26) we have

$$n = -\tfrac{1}{2}(j_e + j_i)(y - \delta) \qquad \text{as} \qquad y \to \delta \quad (32)$$

where $j_{e,\,i} = \Gamma'_{e,\,i}\,r_p/D'_{e,\,i}\,n_\delta'$, and δ is a constant of order

* The author wishes to thank the reviewer for pointing out that the density variation might become important in the certain problem. Consideration of the density variation will be taken into account when we go to the stagnation-probe problem in the next section.

$(\lambda_D/r_p)^{2/3}Re^{1/3}$.[1] Using Eq. (32) in (27) we find the singular behavior as

$$\left(\frac{d\phi}{d\zeta}\right)_v = -\frac{j_e - j_i}{j_e + j_i}\frac{T_w}{\zeta} \qquad \text{as} \qquad \zeta \to 0 \qquad (33)$$

where $\zeta = y - \delta$. The potential equation in term of ζ is

$$-\phi_p = \int_{-\delta}^{\delta_s Re^{1/2}} \left(\frac{d\phi}{d\zeta}\right)_s d\zeta + \int_{\delta_s{}^+Re^{1/2}}^{\delta_v{}^+Re^{1/2}} \left(\frac{d\phi}{d\zeta}\right)_v d\zeta + \int_{\delta_v{}^+}^{\infty} d\mathbf{r}\cdot\nabla\phi_I \qquad (34)$$

where by our definition we can choose $(\lambda_D/r_p)^{3/2}\, Re^{1/3} \ll \delta_s{}^+Re^{1/2} \ll 1$. Because of the singularity shown in Eq. (33), the leading contribution of the right side of Eq. (34) is the coefficient of Eq. (33) multiplied by the logarithm of the thickness of the sheath δ_s. This is easily seen as follows:

$$\int_{\epsilon=\delta_s{}^+Re^{1/2}}^{\delta_v{}^+Re^{1/2}} \left(\frac{d\phi}{d\zeta}\right)_v d\zeta \equiv \int_f \left(\frac{d\psi}{d\zeta}\right)_v d\xi - \left(-\frac{j_e - j_i}{j_e + j_i}\right) T_w \ln\epsilon$$

where the integral on the right is the finite part of the integral on the left, and

$$\int_{-\delta}^{\epsilon} \left(\frac{d\psi}{d\zeta}\right) d\zeta = \int_{-\delta(\lambda_D/r_P)^{-2/3}Re^{1/6}}^{\epsilon(\lambda_D/r_P)^{-2/3}Re^{1/6}} \left(\frac{d\phi}{d\xi}\right)_s d\xi = \int_f \left(\frac{d\phi}{d\eta}\right)_s d\eta + \left(-\frac{j_e - j_i}{j_e - j_e}\right) T_w \ln\epsilon \left(\frac{\lambda_D}{r_p}\right)^{2/3} Re^{-1/6}$$

Thus

$$-\phi_p = -\frac{(j_e - j_i)}{j_e + j_e}\, T_w \ln\left[\left(\frac{\lambda_D}{r_p}\right)^{2/3} Re^{-1/6}\right] + 0(1) \qquad (35)$$

The term designated by 0(1) includes the entire contribution in the inviscid region and the remaining contribution in the viscous and sheath layers.

VI. Stagnation Probe

In the neighborhood of the stagnation point of a body, one can assume that the quantities n, ρ, and T are approximately constant in the inviscid region. The solution of the electric field is then governed by Eq. (13a) subject to the boundary conditions (13b) and (13c). Under the conventional approximation of stagnation flow (flow impinging on an infinite plane), no solution is satisfied by Eq. (13a) subject to the boundary conditions unless 1) $J = 0$, i.e., for a floating body, in which case the electric field in the inviscid region is identically zero; and 2) the body is floating except for a small but finite current element located at the stagnation point. Such an arrangement is of great practical interest. In what follows we shall consider this problem.

We shall make the current element mentioned in the foregoing small enough such that the usual stagnation-flow assumptions can be applied to such an element. However, the element is considered to be much wider than the boundary-layer thickness, so that the electric potential distribution is essentially one-dimensional within the boundary layer. The solutions for the ordinary flow properties are well known. From Eqs. (13) the electric potential in the inviscid region is now given by the following equation and boundary conditions:

$$\nabla^2\phi = 0$$

$$\left.\begin{aligned} \frac{d\phi}{dy_1} &= \frac{aT_\delta}{(D_e' + D_i')n_\delta}\frac{1}{e}\frac{J}{e} \qquad &\text{for} \quad r \le 1 \\ &= 0 & r > 1 \end{aligned}\right\} \text{and } y = 0$$

$$\phi \to 0 \text{ at infinity}$$

where $y = 0$ is the outer edge of the viscous layer, and where the current element is taken to be a circular disk of radius a. We are not very interested in the detailed potential distribution in the half-space $y_i \ge 0$. The only information required for the construction of the current-voltage characteristic is the value of the electric potential on the element $y = 0$ and $r \le 1$.

The solution for the potential is given simply by the potential distribution resulting from a charged disk of surface charge density

$$\sigma = \frac{E_{y0}}{2\pi} = -\frac{1}{2\pi}\frac{JkT_\delta}{N_\delta e^2(D_e' + D_i')} = -\frac{2\lambda_D{}^2}{D_e' + D_i'} J$$

Assuming the variation of the potential across the disk to be small, which assumption is reasonable provided the viscous layer is thin and much smaller than the dimension of the current element, then we need only calculate the potential at the origin. The potential distribution along the y axis is given by

$$\phi_0(y, r = 0) = \int_{\substack{\text{over the}\\ \text{disk}}} \frac{\sigma}{\xi} ds = -\frac{4\pi\, J\lambda_D{}^2}{D_e + D_i}\left[(y^2 + a^2)^{1/2} - y\right] \qquad (36)$$

and $(kT_\delta/e)\phi_\delta(y = 0, r = 0) = -4\pi\,\lambda_D{}^2 aJ/(D_e + D_i)$.

At the floating potential $J = 0$, so that $\phi_\delta = 0$ and $\phi \equiv 0$ in the inviscid region. For $\phi < \phi_f$, $J = J_i - J_e > 0$, and by Eq. (36) we have $\phi_\delta < 0$. The potential in the inviscid region is then negative. For potentials slightly above the floating, we see that $\phi_\delta > 0$. Since the field is zero for the floating potential, the reversal of polarity of the electric field in the inviscid region occurs at the floating potential instead of the plasma potential as in the case of the static plasma.

Within the viscous layer, the flow field is governed by Eqs. (23) with the following assumption[9]:

$$f'^2 - \rho_\delta/\rho \simeq 0\dagger \qquad u^2/H_\delta{}^j \simeq 0$$

$$d(n/\rho)_\delta/d\xi = dH_\delta/d\xi = 0$$

The last two terms in the energy equations of (23) are due to the electron diffusion flux. Under the stagnation-flow assumption, they have a similarity property. The solutions for the velocity and temperature are solved in the usual fashion, i.e., one ignores the existence of the thin sheath by applying the wall conditions for the velocity and temperature to the first and third equations of (23). This procedure is not possible for the second equation of (23) because of the divergence of the electric field. For stagnation flow, Eq (25) is valid throughout the viscous layer. However, as was mentioned before, the constants Γ_e', Γ_i' must be determined by an analysis of the sheath. Once the charged-particle distribution inside the viscous layer is solved, the electric field is then calculated by the use of Eq. (11). Together with the electric field in the sheath, the current-voltage characteristic is given by Eq. (31). A complete solution can be obtained only through a detailed numerical computation (both in the viscous and the sheath layer). However, as indicated in Eq. (35), the leading behavior of the current-voltage characteristic can be obtained analytically if we can solve for the ratio $(j_e - j_i)/(j_e + j_i)$. To give an idea of the accuracy of (35), we take $\lambda_D/r_p = 10^{-5}$, $Re = 10^4$. The relative error by neglecting the terms of order unity is then about 10%. From our definitions of j_e, j_i, and J, we can prove that

$$\frac{j_e - j_i}{j_e + j_i} = \frac{D_i' - D_e'}{D_i' + D_e'} + \frac{2a}{n_\delta' e Re^{1/2}(D_i' + D_e')}\frac{J}{j_i + j_e}$$

Thus, the current-voltage characteristics (35) become

$$\frac{J\, a}{n_\delta' e(D_e' + D_i')} = \frac{1}{2} Re^{1/2}(j_e + j_i) \times \left[\frac{D_e' - D_i'}{D_e' + D_i'} - \frac{\phi_p}{T_w \ln[(a/\lambda_D)^{2/3} Re^{1/6}]}\right] \qquad (3$$

† For a discussion of this approximation, see Ref. 12.

where D_e', D_i' take the values on the surface of the body. From Eq. (26) we see that the quantity $-\frac{1}{2}(j_e + j_i)$ is the charged-particle density gradient near the solid surface given by the quasi-neutral solution. To within an error of the order of the sheath thickness, one can obtain this slope by forcing the density equation to satisfy the boundary condition at the wall. For a complete determination of the density gradient, one has to solve Eqs. (23) simultaneously. However, if we assume that N and Sc are constant throughout the viscous layer, together with the stagnation approximations given earlier, we obtain from the first two equations in (23)[9]

$$z'(0) = 0.47\, Sc^{1/3}/N^{1/2} \tag{38}$$

where N and Sc are evaluated at the wall.

So far we have not considered the density variation across the viscous boundary layer. For the constant stagnation flow, it can be shown from Eq. (22) that $d\eta/dy \sim \rho/\rho_\delta$. At the edge of the sheath $d\eta/dy \sim \rho_s/\rho_\delta$, where ρ_s, ρ_δ are the densities at the edge of sheath and viscous layer, respectively. If this quantity is not of order one, the order of magnitude of the thickness of the sheath and the number density of the charge particles at the edge of the sheath will include the factor ρ_s/ρ_δ. These can be found as follows. From the first expression of (25), we have at the edge of the sheath that $\xi = C_1\, \eta_{\delta_s}$ or

$$\frac{n_s}{n_\delta} = C_1 \frac{\rho_s}{\rho_\delta} \eta_{\delta_s} \tag{39}$$

where η_{δ_s} is the thickness of the sheath in η coordinate. Poisson's equation in the η coordinate is

$$\left(\frac{\lambda_D}{r_p}\right)^2 Re \left(\frac{\rho_s}{\rho_\delta}\right)^2 \frac{d^2\phi}{d\eta^2} = n_e - n_i \tag{40}$$

Within the sheath we introduce $\eta = \eta_{\delta_s}\xi$. Using Eqs. (39) and (40), we find the thickness of the sheath in η coordinate as

$$\eta_{\delta_s} \sim \left(\frac{\lambda_D}{r_p}\right)^{2/3} Re^{1/3} \left(\frac{\rho_s}{\rho_\delta}\right)^{1/3}$$

$$\frac{n_s}{n_\delta} \sim \left(\frac{\lambda_D}{r_p}\right)^{2/3} Re^{1/3} \left(\frac{\rho_s}{\rho_\delta}\right)^{4/3}$$

In order that Eq. (25a) will be valid, one requires that $(\lambda_D/r_p)^{2/3}\, Re^{1/3}(\rho_s/\rho_\delta)^{4/3}$ be small compared with unity. Using $\lambda_D/r_p \sim 10^{-6}$, $Re \sim 10^3 - 10^4$ $\rho_s/\rho_\delta \sim 10 \sim 10^2$, one finds that $n_s/n_\delta \sim 10^{-2} \sim 10^{-1}$.

To find the current-voltage characteristic, we first obtain the value of $j_e + j_i$ through the use of Eqs. (22, 26, and 38), i.e.,

$$-\frac{1}{2}(j_e + j_i) = \frac{0.47\,(Sc)^{1/3}}{N^{1/2}} \left(\frac{\rho_w}{\rho_\delta}\right)^2 2\beta^{1/2}$$

In obtaining this we have used $u_\delta = \beta x$ and set $\rho_s = \rho_w$, the density on the wall. The current-voltage characteristic is then given explicitly as

$$J \frac{a}{n_\delta D_e} = \frac{0.47\, Sc^{1/3}}{N^{1/2}} \left(\frac{\rho_w}{\rho_\delta}\right)^2 (2\beta Re)^{1/2} \times \left\{-1 + \frac{\phi_p}{T_w \ln[(a/\lambda_D)^{2/3} Re^{1/6}(\rho_w/\rho_\delta)^{2/3}]}\right\}$$

where the argument of the natural logarithm is the order of the thickness of the sheath in the unstretched coordinate. The density factor is also included. T_w is the wall temperature normalized by the temperature at the edge of the viscous layer. It should be noted that the preceding formulas are valid only within the floating and the plasma potentials.

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